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BFV–BRST analysis of the classical and quantum q-deformations of the $sl(2)$ algebra

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Abstract. BFV-BRST charge for quantum algebras is not unique. Different constructions of it in classical and quantum phase space for the universal enveloping algebra of the q-deformed $sl(2)$ are discussed. In the quantum framework a positive definite scalar product is used to introduce a co-BFV-BRST charge to study the cohomology problem by means of the techniques of the non-deformed case. Moreover, deformation of the phase space without deforming the generators of $sl(2)$ is considered. $\hbar$-q-deformation of the phase space is shown to yield the Witten's second deformation for $sl(2)$. To study the BFV-BRST cohomology problem when both the quantum phase space and the group are deformed, a two-parameter deformation of $sl(2)$ is proposed, and its BFV-BRST charge is given.

1. Introduction

Gauge symmetries of a Lagrangian manifest themselves as first-class constraints in the Hamiltonian framework. They are in involution on the constraint surface, and in Yang–Mills theories they constitute a Lie algebra after quantization (in the absence of anomalies). These constraints can be employed in constructing a fermionic, nilpotent operator, known as Batalin–Fradkin–Vilkovisky–Becchi–Rouet–Stora–Tyutin (BFV-BRST) charge [1], after quantizing the related phase space and introducing ghost variables (fields). Although ghost variables are an artifact of the quantization procedure, they can be incorporated into classical mechanics by endowing classical phase space with the generalized Poisson brackets. Hence it appears that one can establish a BFV-BRST charge either in a quantum or classical framework [2].

BFV-BRST charges are useful in many aspects. Classical BFV-BRST charge can be employed to find the action which one uses in the path integrals of the underlying theory. Another important aspect is the fact that cohomology classes of BFV-BRST charge are equated with physical states. The importance of the latter property is twofold: (i) one can reveal geometric properties of the theory, (ii) one can utilize the related BFV-BRST charge to obtain a gauge field theory whose solutions of equations of motion coincide with the physical states of the underlying theory.

The above discussion suggests study of the quantum groups [3–7] in a similar manner. Construction of the related BFV-BRST charge is important not only in its own right, because it may elucidate the geometric structure of quantum groups, but also to extract clues useful in formulating related gauge theories. In fact, there have been

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some attempts to formulate a quantum group gauge theory [8, 9], but a complete understanding is lacking.

To keep the resemblance with the usual constrained Hamiltonian systems, we make a distinction between 'quantum groups' realized in a classical phase space endowed with the Poisson brackets, and those realized in terms of commutators after quantization [8, 10]. The former will be noted as 'q-deformation' (or classical q-deformation) and the latter, which is the one commonly called a quantum group, as '$\hbar$-q-deformation' (or quantum q-deformation).

$\hbar$-q-deformed algebras, which are the deformation of Lie algebras, appeared before their classical counterparts. Hence the properties of the former are well established, but to investigate the latter there are different methods which have been proposed recently [8–12].

A powerful way of defining quantum mechanics as $\hbar$-deformation of the classical one is to utilize the Moyal star product [13]. Similarly, a $*$ product is used to acquire q-deformations of classical phase space. Moreover, the $*$ product also establishes the distinction between the $\hbar$- and q-deformations, and it is useful in introducing multi-parameter deformations. Thus to deform the classical phase space we will utilize the $*$ product introduced in [12].

When we deal with the Lie algebras or with the usual constrained Hamiltonian systems, construction of BFV-BRST charge is unique up to canonical transformations. For classical and quantum q-deformed algebras it depends on the differential calculus adopted over the group or on the behaviour of constraints. Some different possibilities are considered in [14–16] for the $\hbar$-q-deformed $sl(2)$ algebra.

One should first answer the question: how many ghost fields are needed? In [14] Woronowicz's deformation of $sl(2)$ is considered and three ghost variables are taken to be one-forms on $sl_{\hbar-q}(2)$. In fact, this seems to be the natural choice when one deals with this realization of $sl_{\hbar-q}(2)$. In [15] $U_{\hbar-q}(sl(2))$ is studied. There also, three ghost variables are used, demanding that the related q-$\hbar$-deformed BFV-BRST charge be a polynomial in q^H. Although this is quite plausible (q^H appears in comultiplication of $U_{\hbar-q}(sl(2))$), it is not the unique choice: the form (or the number) of constraints dictates the number of ghost variables. In [15] it is assumed that there are three constraints behaving as $X_\pm$, and $[H]_q$. But *a priori* one does not know the structure of the constraints. There may be different choices: in [8] a candidate for a quantum group gauge theory is shown to possess infinite gauge field components (hence infinite constraints) depending on the representation of the universal covering algebra. In [16] a '$\hbar$-q-deformation of the BRST algebra' of $sl(2)$ is given by considering the related fields in the fundamental representation. Thus there the number of ghost fields is taken to be four.

We deal with the deformations of $sl(2)$ and choose to work with three ghost variables. As is mentioned above, for $sl_{\hbar-q}(2)$ this is the natural choice. For $U_{\hbar-q}(sl(2))$ this choice makes the comparison of the results with $sl(2)$ explicit. Moreover, it lets one formulate a BFV-BRST charge for a deformation of the $\hbar$-deformed phase space and algebra simultaneously with different parameters (see section 4).

In section 2, two 'q-classical' systems are considered. First, we deal with the phase space endowed with the usual Poisson brackets but a q-deformed 'classical $sl(2)$' algebra. We introduce three ghost fields and discuss two different BFV-BRST charges. One of them is the simplest realization: linear terms in the ghost variables are taken to be linear also in the $sl(2)$ generators, so that there are at most three ghost couplings. The other one possesses five ghost couplings because, one of the linear terms in the ghost variables is taken to be linear in $[H/2]_q$. Then, a $*$ product is defined and used in construction of the 'q-classical mechanics' where the enlarged phase space with three ghost variables is endowed with a new generalized bracket. This formulation is used to obtain a BFV-BRST charge for $sl(2)$ generators.

In section 3, we perform $\hbar$-deformation of the cases studied in section 2. The first of the q-classical systems of section 2 leads to $U_{\hbar-q}(sl(2))$. The BFV-BRST charge which possesses a linear term in $[H/2]_q$ is worked out. We show that in terms of a positive definite scalar product it is possible to introduce a co-BFV-BRST operator whose anticommutator with the BFV-BRST charge gives the quadratic Casimir of $U_{\hbar-q}(sl(2))$. Thus, this approach yields the formulation of the BFV-BRST cohomology of $U_{\hbar-q}(sl(2))$ similar to the usual case [17, 18], which was missing in [15] because there it was supposed that one of the constraints behaves like $[H]_q$. The latter q-classical system after quantization leads to $\hbar$-q-deformation of the phase space and the usual $sl(2)$ generators satisfy Witten's second deformation of $sl(2)$ [6]. We replace the non-vanishing brackets of the ghost variables with the usual anticommutators, but we demand that the anticommutator of the terms of the BFV-BRST charge which are linear in the ghost variables with themselves generates the deformed commutators of $sl_{\hbar-q}(2)$. We find the related BFV-BRST charge and observe that the requirements on the ghost variables lead to the result that they behave like one-forms on $sl_{\hbar-q}(2)$, and the charge as exterior derivative.

In section 4, quantum phase space and $\hbar$-deformed algebra are deformed with separate parameters. Indeed, this is a two-parameter deformation of $sl_{\hbar}(2)$. This two-parameter deformation allows us to write a BFV-BRST operator which interpolates between the BFV-BRST charges of the different realizations of the quantum deformation of $sl(2)$ studied in the previous section.

In section 5 the results obtained, and further perspectives are discussed.

2. Classical BFV-BRST charges

We deal with a one-dimensional system (the usual time coordinate), and $\boldsymbol{R}^2$ phase space. In terms of the phase space variables (p, x), satisfying the usual Poisson brackets

$$\{p, x\} = 1$$

the 'classical $sl(2)$' algebra

$$\{H^0, X^0_\pm\} = \pm 2X^0_\pm, \ \{X^0_+, X^0_-\} = H^0 \tag{1}$$

can be realized if the generators are taken to be

$$H^0 = 2px \qquad X^0_+ = -\sqrt{2}x \qquad X^0_- = \frac{1}{\sqrt{2}}p^2x. \tag{2}$$

We consider 'q-classical systems' defined as:

(1) Poisson brackets are standard, nevertheless the 'classical q-deformed algebra $U_q(sl(2))$' is functionally realized in $C^\infty(\boldsymbol{R}^2)$.

(2) The phase space is endowed with q-deformed Poisson brackets, but the generators are as in (2).

(1) In the phase space endowed with the usual Poisson brackets a functional realization of the 'classical $U_q(sl(2))$'

$$\{H, X_\pm\} = \pm 2X_\pm\,,\ \{X_+, X_-\} = \frac{q^H - q^{-H}}{q - q^{-1}} \equiv [H]_q \tag{3}$$

can be achieved in terms of [10]

$$H = 2px \qquad X_+ = -\sqrt{2}x,\ X_- = \frac{-1 + \cosh(2\alpha p x)}{2\sqrt{2}x\alpha \sinh \alpha} \tag{4}$$

where $q \equiv e^\alpha$.

Let us introduce some ghost variables by enlarging the classical phase space endowed with the generalized Poisson bracket structure, to write a BFV-BRST charge. Let us deal with the cases where there are three ghost variables, but the assumed constraint structures are different from [15].

After choosing three ghost variables and their momenta, we should also define generalized Poisson brackets of them. This depends on the conditions which we require that the BFV-BRST charge satisfies.

To assume that the constraints behave as $X_\pm$, and H, seems to be the simplest choice. By using

$$\begin{aligned}\frac{e^{\alpha H} - e^{-\alpha H}}{e^\alpha - e^{-\alpha}} &= H\frac{\alpha}{e^\alpha - e^{-\alpha}} \sum_{k=0}^{\infty} \frac{(\alpha H)^{2k}}{(2k+1)!}\\ &= H\frac{\alpha}{e^\alpha - e^{-\alpha}} \prod_{k=1}^{\infty}\left(1 + \frac{\alpha^2 H^2}{k^2 \pi^2}\right)\\ &\equiv Hf(H, q)\end{aligned} \tag{5}$$

and introducing the fermionic (ghost) variables (c^i, π_i), $i, j = 0, +, -$, which satisfy the usual generalized Poisson brackets

$$\{\pi_i, c^j\} = \delta_i^j \qquad \{\pi_i, \pi_j\} = 0 \qquad \{c^i, c^j\} = 0 \tag{6}$$

one can write the classical BFV-BRST charge as

$$\Omega_1 = c^+ X_+ + c^- X_- + \frac{1}{\sqrt{2}} c^0 H - \sqrt{2} f(q, H) c^+ c^- \pi_0 + \sqrt{2} c^+ c^0 \pi_+ - \sqrt{2} c^- c^0 \pi_-. \tag{7}$$

The generalized Poisson brackets are

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} + \frac{\partial f}{\partial \pi_i}\frac{\partial g}{\partial c^i} + \frac{\partial f}{\partial c^i}\frac{\partial g}{\partial \pi_i}.$$

One can easily observe that Ω_1 satisfies the classical nilpotency relation

$$\{\Omega_1, \Omega_1\} = 0.$$

We suppose that the generalized Poisson brackets of the ghosts are non-deformed due to the fact that we did not deform the original phase space. But the ghost variables are associated with the gauge (group) generators, so that deforming their Poisson brackets, even if the original phase space is not deformed, is not ruled out.

Another possibility is to suppose that the constraints behave as $X_\pm$, and $[H/2]_q$. The choice where $[H/2]_q$ is replaced by $[H]_q$ seems more natural because in the coproduct of the universal enveloping algebra of q-deformed $sl(2)$, q^H appears. Nevertheless, in the following section we show that our choice possesses more similarities with the usual BFV-BRST cohomology analysis. The related classical BFV-BRST charge satisfying

$$\{\Omega_2, \Omega_2\} = 0$$

is given by

$$\Omega_2 = X_+c^+ + X_-c^- + (q+q^{-1})^{1/2}\left[\frac{H}{2}\right]_q c^0 - (q+q^{-1})^{-1/2}\left(\frac{q^H - q^{-H}}{q^{H/2} - q^{-H/2}}\right)\pi_0 c^+ c^-$$
$$+ \frac{\ln q}{(q-q^{-1})}(q+q^{-1})^{1/2}(q^{H/2} + q^{-H/2})(\pi_+ c^{+c^0 - \pi_-} c^- c^0)$$
$$- \ln^2 q (q+q^{-1})^{1/2}\left[\frac{H}{2}\right]_q \pi_+\pi_- c^- c^+ c^0. \tag{8}$$

(2) As announced before a $*$ product approach is preferred to q-deform phase space (we follow [12]).

Attach a two-dimensional internal space parametrized by ξ, and ρ, to each point of the phase space by defining

$$x_\xi = x\, e^{i\gamma\xi} \qquad p_\rho = p\, e^{i\gamma\rho}. \tag{9}$$

Then define a $*$ product of any functions f and g as

$$f *_\gamma g \equiv \sum_{n=0}^{\infty} \frac{(-\gamma/2)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^k (\partial_\xi^{n-k}\partial_\rho^k f)(\partial_\rho^{n-k}\partial_\xi^k g). \tag{10}$$

this $*$ product is associative and can be used to define the q-deformed Poisson brackets

$$\{f, g\}_0^\gamma \equiv -2\frac{f *_\gamma g - g *_\gamma f}{xp \ln q}$$
$$= \frac{-2}{xp \ln q}\sum_{n=0}^{\infty}\frac{(-\gamma/2)^{2n+1}}{(2n+1)!}\sum_{k=0}^{2n+1}\binom{2n+1}{k}(-1)^k(\partial_\xi^{2n+1-k}\partial_\rho^k f)(\partial_\rho^{2n+1-k}\partial_\xi^k g) \tag{11}$$

where $q \equiv \exp(\gamma^3)$. Let us deal with the functional realization of classical $sl(2)$, given in (4) by replacing $x \to x_\xi$, $p \to p_\rho$:

$$H^\gamma = 2p_\rho *_\gamma x_\xi \qquad X_+^\gamma = -\sqrt{2}x_\xi \qquad X_-^\gamma = \frac{1}{\sqrt{2}} p_\rho * p_\rho * x_\xi.$$

These satisfy the following q-deformed Poisson brackets

$$\{H^\gamma, X_\pm^\gamma\}_0^\gamma = \pm 2AX_\pm^\gamma \qquad \{X_+^\gamma, X_-^\gamma\}_0^\gamma = (A/2)(q^{1/2} + q^{-1/2})H$$

where

$$A = \frac{1-q^{-1}}{\ln q} e^{i\gamma(\xi+\rho)}.$$

It is a 'classical Lie algebra' in terms of the new brackets, thus we obviously need three ghost variables and their momenta for the BFV-BRST analysis. Should the generalized

Poisson brackets of the fermionic ghost variables be deformed or not?† It is completely arbitrary. Hence we suppose that they satisfy the usual conditions

$$c^i c^j = -c^j c^i \qquad \pi_i \pi_j = -\pi_j \pi_i \qquad i \neq j \tag{12}$$

and

$$\{c^i, \pi_j\}^\gamma = \delta^i_j. \tag{13}$$

Then, the generalized q-deformed Poisson brackets are

$$\{f, g\}^\gamma \equiv -2\frac{f *_\gamma g - (-)^{\varepsilon(f)\varepsilon(g)} g *_\gamma f}{xp \ln q} + \frac{\partial_l f}{\partial \pi_i}\frac{\partial_r g}{\partial c^i} - (-)^{\varepsilon(f)\varepsilon(g)}\frac{\partial_l g}{\partial \pi_i}\frac{\partial_r f}{\partial c^i} \tag{14}$$

where $\varepsilon(f)$ indicates the ghost number:

$$\varepsilon(c^i) = -\varepsilon(\pi_i) = 1 \qquad \varepsilon(fg) = \varepsilon(f) + \varepsilon(g).$$

Hence we write the BFV-BRST charge as

$$\Omega_3 = H^\gamma c^0 + X^\gamma_+ c^+ + X^\gamma_- c^- - A\pi_+ c^0 c^+ + A\pi_- c^0 c^- - (A/2)(q^{1/2} + q^{-1/2})\pi_0 c^+ c^- \tag{15}$$

which satisfies

$$\{\Omega_3, \Omega_3\}^\gamma = 0.$$

One can observe that if we keep (13) but deform (12), a BFV-BRST charge which possesses terms linear in the generators as in (15), will not exist.

3. Quantization

When we deal with the non-deformed phase space, there is no difference between introducing the $\hbar$-deformation in terms of the Moyal brackets or canonical quantization as far as the purposes of this section are considered. If we drop $*$ in the former formulation, both of them will yield the following fundamental commutators

$$[p, x] = -i\hbar. \tag{16}$$

Of course, when (16) is considered as Moyal brackets p and x are classical variables, but they are operators in terms of the canonical quantization.

After an appropriate rescaling of the generators, and replacing the Poisson brackets with commutators, (1) becomes the usual $sl(2)$ algebra and (3) yields $U_{\hbar-q}(sl(2))$ [10]:

$$[H, X_\pm] = \pm 2X_\pm, \ [X_+, X_-] = [H]_q. \tag{17}$$

The ghost fields, then, satisfy

$$[\pi_i, c^j] = -i\hbar\delta^j_i$$

where $[f, g] = fg - (-)^{\varepsilon(f)\varepsilon(g)} gf$. For simplicity we rescale the phase space variables such that

$$[p, x] = 1 \qquad [\pi_i, c^j] = \delta^j_i.$$

† In the $q-\hbar$-deformed case there is somehow a natural answer to this. See section 3.

Under the $\hbar$-deformation $\Omega_1 \to Q_1$ which is in the same form but satisfying

$$Q_1^2 = 0. \tag{18}$$

The BFV-BRST charge for (17), Q_2, satisfying $Q_2^2=0$, when the constraints are supposed to behave like $X_\pm$, and $[H/2]$, is no longer similar to (8), but

$$\begin{aligned} Q_2 = {} & X_+ c^+ + X_- c^- + (q+q^{-1})^{1/2}\left[\frac{H}{2}\right]_q c^0 - (q+q^{-1})^{-1/2}\frac{q^H - q^{-H}}{q^{H/2}-q^{-H/2}}\pi_0 c^+ c^- + \frac{(q+q^{-1})^{1/2}}{q^{1/2}-q^{-1/2}} \\ & \times \{(q^{(H+1)/2}+q^{-(H+1)/2})\pi_+ c^+ c^0 - (q^{(H-1)/2}+q^{(-H+1)/2})\pi_- c^- c^0)\} \\ & - \frac{(q+q^{-1})^{1/2}}{(q^{1/2}+q^{-1/2})^2}(q-q^{-1})^2\left[\frac{H}{2}\right]_q \pi_+\pi_- c^- c^+ c^0. \end{aligned} \tag{19}$$

To obtain the physical states or the solution of the BRST cohomology, let us consider the space of the states

$$\Psi(c) = \sum_{l=0}^{3} \frac{1}{l!} c^{i_1}\cdot c^{i_l}\Psi^{(l)}_{i_1\cdot i_l}.$$

The $\Psi^{(l)}$ coefficients are some complex functions on the space where the constraints or the generators act. Action of π_i on the states is

$$(\pi_i\Psi)^{(l)}_{i_1\cdot i_l} = \Psi^{(l+1)}_{i i_1 i_l} \qquad l=0,1,2.$$

When one deals with a Lie algebra the coefficients $\Psi^{(l)}_{i_1\cdot i_l}$, can be considered as l-forms on the algebra, and the indices are raised or lowered by the Cartan metric of the algebra. Thus one can introduce the scalar product [18]

$$(\Phi, \Psi) = \sum_{l=0}^{3} \frac{1}{l!}\Phi^{\dagger(l) i_1\cdot i_l}\Psi^{(l)}_{i_1\cdot i_l} \tag{20}$$

which is positive definite. With respect to this product

$$c^{i\dagger} = \pi_i. \tag{21}$$

$Q_L^\dagger$ obtained from the BFV-BRST charge Q_L of the Lie group is also nilpotent. When we deal with $sl(2)$ and demand that $[Q_L, Q_L^\dagger]$ is a generalization of the quadratic Casimir of the algebra, in the basis we adopted, the scalar product should also yield

$$X_\pm^\dagger = X_\pm \qquad H^\dagger = H. \tag{22}$$

In the case where we assume that the constraints behave like $X_\pm$, H, the conjugation defined by (21) and (22) leads to $Q_1^\dagger$, which is nilpotent. Unfortunately, when the constraints are supposed to behave like $X_\pm$, and $[H/2]$, $Q_2^\dagger$ obtained from (19) is not nilpotent. This is due to the fact that in the former case BFV-BRST charge is insensible to the ordering of ghost variables, but in the latter a change in the ordering of ghost variables would create some terms which spoil the nilpotency condition.

To overcome this difficulty let us introduce the following positive definite scalar product

$$(\Phi^*, \Psi) = \sum_{l=0}^{3} \frac{1}{l!}\Phi^{*(l)}_{i_l\cdot i_1} g^{i_1 j_1}\cdot g^{i_l j_l}\Psi^{(l)}_{j_1\cdot j_l} \tag{23}$$

where $g^{00} = g^{+-} = g^{-+} = 1$, and $\Phi^{*(l)}_{i_l\cdot i_1}$ is the complex conjugate of $\Phi^{(l)}_{i_l\cdot i_1}$. With respect to this product the conjugate of the generators and the ghost variables are

$$X_\pm^* = X_\pm \qquad H^* = H \qquad c^{0*} = \pi_0 \qquad c^{+*} = \pi_- \qquad c^{-*} = \pi_+ \tag{24}$$

where $(f^*)^* = f$. Conjugation of Q_1 yields the following co-BFV-BRST charge

$$Q_1^* = \pi_- X_+ + \pi_+ X_- + \frac{1}{\sqrt{2}} \pi_0 H - \sqrt{2} f(q, H) \pi_- \pi_+ c^0 + \sqrt{2} \pi_- \pi_0 c^- - \sqrt{2} \pi_+ \pi_0 c^+ \tag{25}$$

which is nilpotent and $[Q_1, Q_1^*]$ is a generalization of the quadratic Casimir of $sl(2)$. This justifies the choice of the normalization factors of the terms linear in the ghost variables.

In terms of the conjugation given in (24), the co-BFV-BRST charge derived from Q_2 is

$$\begin{aligned} Q_2^* = {} & X_+ \pi_- + X_- \pi_+ + (q+q^{-1})^{1/2} \left[\frac{H}{2}\right]_q \pi_0 \\ & - (q+q^{-1})^{-1/2} \frac{q^H - q^{-H}}{q^{H/2} - q^{-H/2}} c^0 \pi_- \pi_+ + \frac{(q+q^{-1})^{1/2}}{q^{1/2} - q^{-1/2}} \\ & \times \{ (q^{(H+1)/2} + q^{-(H+1)/2}) c^- \pi_- \pi_0 - (q^{(H-1)/2} + q^{(-H+1)/2}) c^+ \pi_+ \pi_0) \} \\ & - \frac{(q+q^{-1})^{1/2}}{(q^{1/2} + q^{-1/2})^2} (q - q^{-1})^2 \left[\frac{H}{2}\right]_q c^- c^+ \pi^+ \pi^- \pi^0. \end{aligned} \tag{26}$$

This nilpotent charge, as in the usual case, can be used to define

$$[Q_2, Q_2^*]|_{\pi = c = 0} = C_q \tag{27}$$

where C_q is the quadratic Casimir of $U_{\hbar-q}(sl(2))$ [19, 20]:

$$C_q = X_- X_+ + \left[\frac{H}{2}\right]_q \left[\frac{H+2}{2}\right]_q = \frac{1}{2}\left(X_- X_+ + X_+ X_- + (q+q^{-1}) \left[\frac{H}{2}\right]_q \left[\frac{H}{2}\right]_q \right).$$

Hence by using the positive definite scalar product (23), the physical states can be identified with the states ω satisfying

$$(Q + Q^*)^2 \omega = 0 \qquad Q\omega = 0 \qquad Q^* \omega = 0$$

where Q and Q^* are given either by (18) and (25) or by (19) and (26). At zero ghost number the cohomology classes given by (19) include the ones found in [15], and the states ω_2 satisfying $Q_2 \omega_2 = 0$, contain the singlets of $U_{\hbar-q}(sl(2))$. Although at zero ghost number $Q_1 \omega_1 = 0$ yields the states ω_1, which are singlets of $sl(2)$, by including ghost number one sector the other states of $U_{\hbar-q}(sl(2))$ can be obtained.

If we $\hbar$-deform the phase space after the q-deformation we obtain [12]

$$x *_{\gamma\hbar} p - qp *_{\gamma\hbar} x = -i\hbar q^{1/2}. \tag{28}$$

The $*_{\gamma\hbar}$ product is defined as

$$\begin{aligned} & f(x,p) *_{\gamma\hbar} g(x,p) \\ & \quad = f(x_\xi, p_\rho) \sum_{n=0}^{\infty} \frac{(-\gamma/2)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} (-)^k (\partial_\xi^{n-k} \partial_\rho^k)(\partial_\rho^{n-k} \partial_\xi^k) \\ & \quad \times \sum_{m=0}^{\infty} \frac{(-i\hbar/2)^m}{m!} \sum_{l=0}^{m} \binom{m}{l} (-)^l (\partial_x^{m-l} \partial_p^l)(\partial_p^{m-l} \partial_x^l) g(x_\xi, p_\rho)|_{\xi = \rho = 0} \end{aligned}$$

where in the sums the first two derivatives act on the left and the others on the right-hand side.

For our purposes, once the $\hbar$-deformation is achieved we can forget about the $*_{\gamma\hbar}$ and also set $\hbar = 1$, to obtain

$$xp - qpx = -\mathrm{i}q^{1/2}. \tag{29}$$

By keeping the form of the generators as in (2) we obtain

$$HX_- - qX_-H = -\mathrm{i}2q^{1/2}X_- \tag{30}$$

$$HX_+ - q^{-1}X_+H = \mathrm{i}2q^{-1/2}X_+ \tag{31}$$

$$X_+X_- - q^2X_-X_+ = (-\mathrm{i}/2)(q^{1/2} + q^{3/2})H. \tag{32}$$

After rescaling as

$$X_\pm \to \mathrm{i}q^{-1/2}(q^{1/2} + q^{3/2})^{1/2} \qquad H \to \mathrm{i}2H$$

and setting

$$q = r^2$$

one can see that the relations given in (30)–(32) read

$$\begin{aligned} &r^{-1}HX_- - rX_-H = -X_- \\ &rHX_+ - r^{-1}X_+H = X_+ \\ &r^{-2}X_+X_- - r^2X_-X_+ = H \end{aligned} \tag{33}$$

in which we recognize Witten's second deformation [6].

When we q-$\hbar$-deform the phase space a natural requirement for the BFV-BRST charge is to demand that the anticommutation of the terms which are linear in ghost variables and the generators with themselves generates the deformed commutators of the algebra (left-hand side of (33)). For the deformed algebra (33), this condition leads to

$$c^ic^j = -v^{ij}c^jc^i \qquad v^{ii} = 0 \qquad v^{0+} = r^2 \qquad v^{-0} = r^2 \qquad v^{-+} = r^4. \tag{34}$$

These relations could also be obtained by demanding that c^i behave like one-forms on $sl_{\hbar-q}(2)$ [5].

Although one can also deform the commutators, it is not necessary. In fact, we deal with the ghost variables satisfying (no summation over i)

$$[\pi_i, c^i] = \delta_i^i \qquad c^{i2} = \pi_i^2 = 0. \tag{35}$$

Now the associativity leads to

$$\pi_i\pi_j = -v^{ij}\pi_j\pi_i \qquad \pi_ic^j = -v^{ji}c^j\pi_i.$$

Hence the BFV-BRST charge which satisfies $Q_3^2 = 0$, is

$$Q_3 = Hc^0 + X_+c^+ + X_-c^- - r(\pi_+c^+c^0 + \pi_-c^0c^-) - r^2\pi_0c^+c^- - (r - r^{-1})\pi_-\pi_+c^0c^+c^-. \tag{36}$$

One can observe that the choice (34), (35) follows if we require that Q_3 behaves like the exterior derivative, so that $[Q_3, c^i]$ coincide with the Cartan–Maurer structure equations on $sl_{\hbar-q}(2)$.

To find solution of the cohomology of Q_3 one should define a state space endowed with a scalar product, and introduce the co-BFV-BRST charge. A choice is given in [14]. The choice should be dictated by the desired physical content of the gauge theory. This is still obscure to us, so that the issue of defining a scalar product and co-BFV-BRST charge is not discussed here.

4. Two-parameter deformation

In the procedure which we follow, the next step is obviously to deform the $\hbar$-deformed phase space as well as the $\hbar$-deformed algebra with different parameters. This will lead to a two-parameter deformation of $sl_\hbar(2)$.

Recently, attention has been paid to multiparameter deformations of Lie groups [21]. Because of the fact that the requirements are different, not all of these deformations fulfil the condition of being a Hopf algebra. Indeed, the two-parameter deformation which we introduce below does not seem to be Hopf algebra for all values of the parameters.

In this section deformation of the quantum phase space is supposed to be realized as given in [4], which is known to be equivalent to Witten's deformation (33) [20], and deformation of the algebra is given in (17). If we demand to obtain one of the deformations at some special values of the deformation parameters, it is quite natural to consider the following two parameter deformation of $sl_\hbar(2)$,

$$\begin{aligned}
&\mu^2 HX_+ - \frac{1}{\mu^2} X_+H = (1+\mu^2)X_+ \\
&\mu^2 X_-H - \frac{1}{\mu^2} HX_- = (1+\mu^2)X_- \\
&\frac{1}{\mu} X_+X_- - \mu X_-X_+ = [H]_q
\end{aligned} \tag{37}$$

which can be noted as $U_q(sl_{\hbar-\mu}(2))$.

To keep the resemblance with $\mu=1$ and $q=1$ cases we introduce the ghost fields satisfying (no summation over i)

$$[\bar\eta_i, \eta^i] = \delta_i^i \qquad \bar\eta_i^2 = \eta^{i2} = 0 \tag{38}$$

$$\eta^i\eta^j = l_{ij}\eta^j\eta^i \qquad \bar\eta_i\bar\eta_j = l_{ij}\bar\eta_j\bar\eta_i \qquad \eta^i\bar\eta_j = l_{ji}\bar\eta_j\eta^i \tag{39}$$

where

$$l_{0+} = \mu^4 \qquad l_{-0} = \mu^4 \qquad l_{-+} = \mu^2.$$

The BFV-BRST charge which leads to the one given in [14] for $q=1$, and to the one given in [15] for $\mu=1$, moreover satisfying $Q^2=0$, is

$$\begin{aligned}
Q = {} & X_-\eta^- + X_+\eta^+ + [H]_q\eta^0 - \mu\,\bar\eta_0\eta^+\eta^- - F(H)\,\bar\eta_+\eta^+\eta^0 \\
& + G(H)\bar\eta_-\eta^0\eta^- - (G(H)+F(H))\bar\eta_-\bar\eta_+\eta^+\eta^0\eta^-
\end{aligned} \tag{40}$$

where

$$F(H) = \frac{\mu^2}{q-q^{-1}}\,[\mu^{-2}(q^H - q^{-H}) - \mu^2(q^a - q^{-a})]$$

$$G(H) = \frac{\mu^2}{q-q^{-1}}\,[\mu^2(q^H - q^{-H}) - \mu^{-2}(q^b - q^{-b})]$$

$$a=\mu^{-2}(\mu^{-2}H+\mu^2+1)$$

$$b=\mu^2(\mu^2H-\mu^2-1).$$

The form of this BFV-BRST charge in ghost variables is the same with (19) and (36). This follows from the fact that non-vanishing commutators between the ghost variables in all the cases are kept non-deformed.

To find solutions of the Q-cohomology, one should introduce a state space and a scalar product. Obviously, there are different choices and as is mentioned above, it is closely related to the desired physical properties of the system.

5. Discussion

Let us consider the results achieved for 'classical $U_q(sl(2))$' and the deformation of phase space separately. In the former case we have shown that different BFV-BRST charges can be written, and have observed that the BFV-BRST charges obtained in the classical and in the quantum framework are very similar. Moreover, the realization given in (19) by making use of the positive definite scalar product (23), allows us to formulate the cohomology problem of the deformed algebra, similar to the usual one given in [17, 18], which was missing in [15].

Generators of $sl(2)$ in q-deformed phase space still satisfy a 'classical Lie algebra' in terms of the new brackets, so that the related BFV-BRST charge (15) possesses at most three ghost couplings. In $\hbar$-q-deformed phase space, it is shown that the generators of $sl(2)$ satisfy the relations of $sl_{\hbar-q}(2)$, (30)–(32), and the BFV-BRST charge possesses also a five-ghost coupling term.

The desire to incorporate the two different approaches to quantum groups led us to define a two-parameter deformation of $sl(2)$. Because of not deforming the non-vanishing generalized commutators of the ghosts we were able to find a BFV-BRST charge (40) which interpolates between the BFV-BRST charges of the different realizations of the one-parameter deformation of $sl(2)$.

There are mainly two future perspectives regarding the BFV-BRST charges presented here: (i) to study their cohomology problem, and (ii) to show if they can be used to define a BRST field theory which leads to a gauge theory of q- or $\hbar$-q-deformed $sl(2)$.

The properties of the two-parameter deformation of $sl(2)$ presented in section 4, are unknown. It would be useful to understand the algebraic features of it, to incorporate the different realizations of the quantum q-deformed $sl(2)$.

Here, we considered only a one-dimensional system and R^2 phase space. To obtain a more realistic system, one should study either a higher dimensional system or a larger phase space. The former can be useful for studying gauge field theories whose gauge group is a quantum group. Alternatively, the latter can be used to define a gauge theory of quantum groups by means of BRST gauge theory.

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